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Ranking the Components of Algebraic Thinking Model in Sixth-Grade Students during Coronavirus Pandemic Based on Fuzzy Analysis

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ABSTRACT

Objective: Studies on teaching and learning algebra point to the fact that algebra is a branch of mathematics that deals with the general notation of relationships between numbers and mathematical structures and also performing operations on those structures. Algebra problems that students face is mainly in mathematical modeling, perception of algebraic expressions, applying arithmetic operations, and comprehending the different meanings of the equal sign and the variable symbol. Given the lack of attention to the promotion of the algebraic thinking components, the current study aimed to explain and prioritize these components in sixth-grade elementary students through virtual education during the coronavirus pandemic.

Methods: To collect data based on survey research, an open and closed-ended questionnaire was used.

Results: Statistical analysis revealed that the students mostly achieved generalized arithmetic (2.92) and modeling in algebraic thinking (2.89), and they tended to answer through generalization and relied on pre-learned formulas. The lowest dispersion of answers was related to the reasoning component. Fuzzy analysis indicated the modeling component was ranked first (0.075251). And the reasoning component was ranked fourth (0.074962). The third and second ranks respectively belonged to the functions and generalized arithmetic component.

Conclusions: Consequently, ranking algebraic thinking components could be effective for teaching mathematics.

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Introduction

Learning mathematics is one of the valuable achievements in the field of school education which can be considered one of the fundamental components of the school curriculum as well. According to the outcomes of Trends in International Mathematics and Science Study (Timss), challenges and problems of teachers and students in the process of teaching and learning mathematics have been observed all over the world, especially in Iran. This is a problem that we can view its effects in the future generation and the economic-industrial developments of the country. Whenever the topic of education and learning is discussed in every country, we encounter challenges in some content and abstract concepts such as Math, physics, etc. Therefore, investigation and research in the field of teaching and learning mathematics is essential and inevitable (Eisner, 2000). Students learn either more or less than what they are taught in the classroom. Students create and discuss meanings and concepts that are an achievement of the materials taught by the teachers. Research on teaching and learning algebra refers to the fact that it is a branch of mathematics that deals with the general symbolization of relationships between numbers and mathematical structures and performing operations on those structures. Since we discuss algebra in school, we sometimes refer to school algebra as generalized arithmetic. In other words, school algebra means writing the general modes of presentation of arithmetic rules and operations (Kazemi moghadam, 2017). Kilpatrik and Soford (2002) believed that algebra is a gateway to higher-level mathematics and a visualization of the relationships between numbers. Algebraic expressions provide numerous functions. For instance, when the Pythagorean Theorem is applied to the right triangle, it is actually an equality between two algebraic expressions which we use in geometric calculations. In a standard document, the National Council of Mathematics Teachers of America (NCTM, 2000) stated that all students in all educational systems should learn algebra from elementary to the last year of high school. Regarding school algebra, the council believes that educational programs from kindergarten to the end of high school should assist students to learn patterns, relationships, modeling, and functions and to describe and analyze mathematical structures utilizing algebraic symbols among numbers.

Algebra is a part of mathematics (Nurlaeli & Agoestanto, A. Mashuri, 2018). Mathematicians describe algebra as the ability to utilize and recognize the function of variables as the relationship between known and unknown variables. Driscoll (1999) noted that algebraic ability is the skill to

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present forms quantitatively in such a way that the relationship between variables becomes explicit (Choudhury & kumar Das, 2012). Thus, the algebraic ability can be illuminated as a person's ability to clarify algebraic perception as an aspect of relation, abstraction, and different forms of calculation. Earlier studies in this field pointed out students' problems in algebraic thinking. These studies revealed that some of the problems that students encounter in algebraic abilities include misunderstandings/lack of correct calculation in completing and solving algebraic operations (Mashuri et al., 2018; Misbahuddin et al., 2019).

Some factors that cause difficulty for students to learn mathematics, particularly algebra are personal factors (both intrapersonal and extra-personal), learning aspects, and factors related to students' concentration and notice to learning mathematics. Students with high mathematical abilities can think algebraically in rewriting information mathematically. They think by expressing the relationships in a pattern or rules that are usually used in a problem through representations in the form of algebra, images, and words. And using these factors, they can apply and interpret mathematical findings. They can use rules or patterns to provide solutions for any algebraic problem (Warsitasari, 2015).

Consequently, algebra is at a high level of mathematical knowledge. In learning mathematics, the concept of algebra is a generalization of arithmetic. Thus, algebra plays a significant role in mathematics. Students still face difficulties in learning algebra. Algebraic problems that students face are predominately in mathematical modeling, understanding algebraic expressions, applying arithmetic operations, and perception of the different meanings of the equal sign and the variable symbol (Jupri et al., 2014). Most learners have problems in generalizing arithmetic applying algebraic symbols (Ayber & Tanışlı, 2017). In algebraic thinking, for changes from one algebraic form to another, students' cognitive development is more significant than calculation and algebraic operations (Loibl & Leuders, 2019). Students prefer to answer verbally instead of symbolic expression (Zayyadi et al., 2019). Teachers should train algebraic reasoning abilities to overcome students' problems.

Algebraic thinking is the most frequent way of formal algebra learning, and algebra greatly occupies fields of mathematics (Jacobs et al., 2007; Rittle-Johnson et al., 2011). Literature has revealed that students who learn algebra well in mathematics have more opportunities to attend technical and engineering courses in universities (Kim et al., 2015). Even though, many students

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cannot understand algebraic concepts and succeed in learning, and middle school students have difficulty in learning algebra due to the abstractness of its subject (Kaput et al., 2017). Algebra is regarded abstract since arithmetic and algebra are considered two separate subjects in school. And therefore, they are taught as separate subjects in many school curricula. Arithmetic entails the direct use of numbers while algebra refers to the use of letters to represent numbers. In this way, a cognitive gap appears in the transition from arithmetic to algebra (Kızıltoprak & Köse, 2017). This problem can be solved by developing algebraic thinking during teaching arithmetic in elementary school. Since algebra is not a distinct entity from arithmetic, teachers can instruct students to think algebraically while learning arithmetic (Blanton & Kaput, 2004). Algebra is not a subject that is exclusively taught in high school. It is often associated with symbols and variables, though algebraic thinking is not always the case.

Teachers believe that algebra is one of the most crucial branches of mathematics. The students' learning difficulties are related to the meaning of letters, changing from arithmetic to algebraic conventions, and recognizing and using structures. Some corresponding research indicated that students make mistakes in solving problems related to algebraic operations such as variables, negative signs, solving algebraic equations, and solving fractions. The learners' difficulty in algebraic thinking are in interpreting the information of the problem in mathematical language, understanding the received information and the question in the problem, and the combination of logic and concept that they have learned while solving the problem. By analyzing students' answers in solving problems algebraically, it is argued that learners have difficulty in clarifying and describing quantitative relationships, especially those that involve proportional relations. Also, they encounter trouble in getting information from given questions. Therefore, learners have trouble in predicting patterns, dividing information and generalizing rules in operations, and students are not compatible with algebra in general (Andini & Suryadi, 2017).

Algebra learning in school initiates the elementary levels. Most teachers are unaware of the algebraic nature of the concepts as well as the relationships between numbers. In their view, algebra begins after elementary school. Expressions such as; $+5=10$ in the first and second grades of elementary school (see Figure 1) is the beginning of algebra concept in mathematical knowledge. Such expressions appear in other grades of elementary school. So that students in the sixth grade (see Figures 2 and 3) observe such expressions in fractional forms, completing the relationship between figure number and matchsticks number, etc. Paying attention to the category of algebra in elementary school can lead to the development of mathematical thinking in higher grades as well as problem-solving in other sciences such as physics.

Since the beginning of 2020, learner's physical attendance in all schools, educational centers, and universities was changed due to the covid-19 pandemic. Thus, conventional face-to-face education changed and it caused a revolution in online education. Migration from the physical learning environments to virtual-online environments became a compulsory task for all teachers and trainers. There were various responses to emergency condition around the world. Educational environments were launched to provide platforms and models for online learning at home for students. In some situations, educational professionals adapted their models and supported systems based on their local needs (Tian, Yao, & Ding, 2020). Researchers intended to investigate what people have learned or aimed to learn in the pandemic situation with non-attendance training (Bakker & Wagner, 2020). They investigated the application of technologies in teaching math (Clark-Wilson et al., 2020), reviewed classrooms with online-offline platforms in different styles, and indicated their strategies, advantages, and challenges. However, the method of online teaching mathematics in a hybrid model, both synchronously and asynchronously, is still a new field (Di Pietro et al., 2020; Ferdig et.al, 2020). With the provision of the internet and technologies, public-private teaching systems and student self-regulated training can be presented (Engelbrecht et al., 2020).

Figure 1. An example of expressions of the basic algebra concept in the second-grade elementary school in operational form

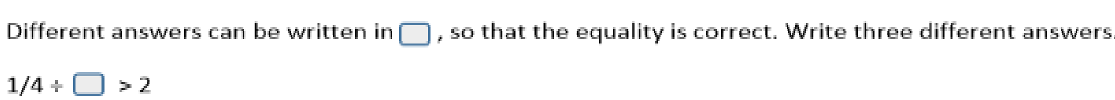

Figure 2. An example of expressions of the basic algebra concept in the sixth-grade elementary school in an unequal form

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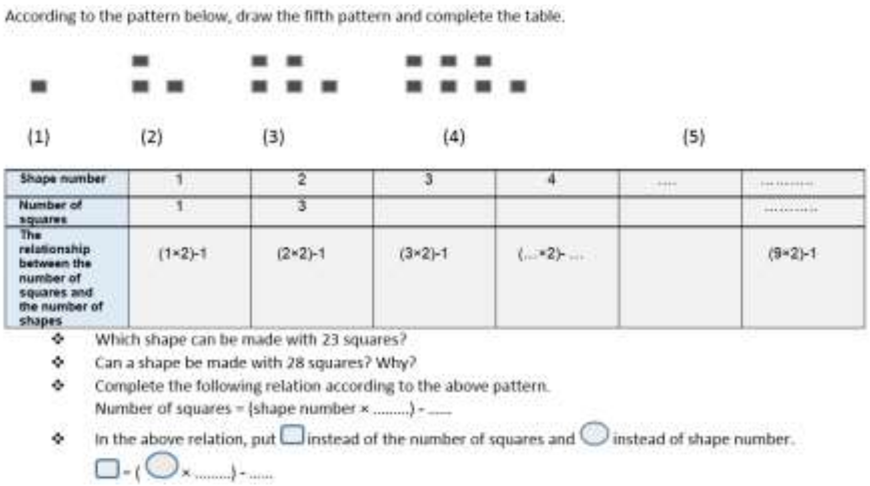

According to the pattern below, draw the fifth pattern and complete the table.

(1) (2) (3) (4) (5)

Shape number	1	2	3	4	.....	....
Number of squares	1	3				....
The relationship between the number of squares and the number of shapes	(1×2)-1	(2×2)-1	(3×2)-1	(...×2)- ...		(9×2)-1

- Which shape can be made with 23 squares?
- Can a shape be made with 28 squares? Why?
- Complete the following relation according to the above pattern.
  Number of squares = (shape number × ....) - ......
- In the above relation, put ▢ instead of the number of squares and ◯ instead of shape number.
  ▢ = (◯ × ....) - ......

Figure 3. An example of numerical relationships and making a model of the basic algebra concepts in the sixth-grade elementary school

During the coronavirus pandemic in Iran, the training in schools was changed to non-attendance and the challenges of teaching and learning in schools were obvious. Attention to thinking, problem-solving, and creative ideas while solving math and science problems in elementary school required special attention. The current research was also reviewed online in the same training course to examine the challenges, advantages, and problems of students' learning, especially in the field of algebra and the type of thinking. Therefore, teachers must evaluate the components of algebraic thinking in students at any level in the special education course. In order to carry out such an assessment continuously, it was required to design and validate an instrument for measuring algebraic thinking. In the current study, an attempt was made to answer this research question: which component of the algebraic thinking model of sixth- grade students is the priority? The nature of mathematical science is the knowledge and study of the order and organization of numbers, and its subject is the clarification and explanation of the order that is hidden in different situations. The fundamental tools of knowledge are the concepts that enable people to describe and examine the order. From this point of view, we can consider mathematics as an essential requirement in schools (Baloglu & Kocak, 2006; Dehaene et al., 1999) Contrary to some ideas, mathematical knowledge is not a set of prepared and predetermined formulas/principles, but it is the understanding of the problem through which one can solve it. To gain the ability to acquire mathematical knowledge, students should be able to allocate appropriate time to solve a problem and finally answer it through thinking, reflection, and creativity (Clausen-May, 2005). Therefore, we can claim that mathematical knowledge is the skill of solving problems, examining numbers, shapes, and objects, and establishing the required proportions in all applied sciences (Reyna & Brainerd, 2007). School mathematics involves the education of students who can solve problems logically, to provide a reason, and to analyze in dealing with various issues of professional and personal life (Hakkarainen et al., 2013).

Algebra is widely recognized as a gateway to higher education and job opportunities. Various descriptions of algebra can be found in different contexts. For example, Usiskin (1988) presented four concepts of school algebra: Algebra as generalized arithmetic ($a + b = b + a$), Algebra as the study of procedures for solving certain types of problems (5x+3=40), Algebra as the study of the relationship between quantities (y=11x+b), Algebra as the study of structures ($3x^2 + 4ax - 132a^2$) Kaput (1995) classified algebra based on five features: a) generalization and formalization; b) guided syntactic manipulations; c) studying structure; d) studying functions, relationships, and common changes; e) Modeling language. According to him, generalization, formalization, and syntactic manipulation are the factors that underlie all the others. Kieran (1981) classified school algebra based on students' activities: a) generational activities, b) transformative activities, and c) general supra-level activities. Developing expressions and equations to represent problem situations or generalizations refers to general activities.

Law-based activities such as the collection of terms, factoring, and simplifying refer to transformative activities. An important aspect of this activity is to maintain equivalence despite deformation. At last, the activities of problem-solving, modeling, and proof, in which algebra plays a role as a tool, are considered extra-level activities. However, Lee (2001) described algebra as follows: Algebra is like a language; in other words; algebra is as a way of thinking, as a problem-solving activity, as a tool to make thinking and conveying messages more effective, and algebra as a generalized arithmetic.

Algebraic thinking frequently includes the process of generalizing mathematical procedures and, as it becomes more complex, it deals with unknown numbers. Considering pattern recognition and mathematical generalization, teachers should attentively guide students to algebraic thinking. The process of changing from real or mathematical contexts to structures is known as algebraic thinking. Developing an individual’s abilities to understand and apply symbols is part of the process. As the students begin to generalize mathematical knowledge and numerical patterns, their

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algebraic thinking develop. Symbolization and good generalization abilities are needed for strong algebraic reasoning. Kaput et al. (2017) stated that algebra is a cultural product or a body of knowledge that is rooted in educational institutions. Mathematical operations are implemented syntactically on symbols in classical symbol systems. Generalization, transformation, and transformational algebra are three branches of school algebra. Algebraic thinking is a way of approaching mathematical problems that focuses on the significance of general relationships between numbers. Algebraic thinking includes the process of generalizing arithmetic operations and, as it becomes more complex, it deals with unknown quantities. Five categories of algebraic thinking are as follows: Generalization and formulation of arithmetic operations, Manipulation and transformation of definite equality problems through inverse operations and original composition, Analysis of mathematical structures, Relations and functions, including numbers and letters, Algebraic language and representation (Schliemann, 2013; Stephens et al., 2015).

Teachers should conscientiously instruct students concerning pattern recognition and mathematical generalization to algebraic thinking, since they acquire arithmetic skills (Carraher et al., 2000). For instance, students explore the property of identity by investigating the objective equation $5 \times 1 = 5$ in different values. Then, they realize the value of each number that is multiplied by a number, the pattern leads to the rule that a number preserves its identity. In the end, they learn to generalize in the form $a \times 1 = a$ using the same letter as the symbolic representation of the same number of each value and so on (Lentz, 2018). Carpenter and Levi (2000) defined algebraic thinking as a) generalization and b) using symbols to represent mathematical ideas, and to represent and solve problems. Kaput et al. (2017) concentrated on Carpenter and Levi (2000) definition of algebraic thinking as a conceptual framework used in studies. They conceptualized algebraic thinking as a generalization and symbol of two different aspects, including three stages: generalized arithmetic, functions, and modeling (Kaput et al., 2017). The symbolic aspect was described as "the generalized systematic symbolization of arrangements and constraints", while the generalization aspect was described as "compositional reasoning and actions related to generalization expressed in conventional symbolic systems." The model adapted for this specific research on elementary school students is revealed in Figure 4. Simulation and generalization are required in all three stages, but generalization may be more usual in generalized arithmetic, while representation may be more frequent in modeling studies (Ralston, 2013).

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The reasoning is a thought process that arises from empirical observations in which several new concepts and deep understandings are created. Mathematical reasoning is a habit of thinking, and like a habit, the reasoning is a constant part of each student's mathematical experience. The NCTM (2000) revealed that mathematics is a reasoning science that any activity in mathematics cannot be separated from reasoning. Therefore, reasoning becomes a greatly important basic ability to improve mathematical skills. The reasoning ability of students in the majority of elementary school classes is low. The issue has mainly been created by teachers who are still tended towards procedural and technical subjects and they teach mathematical concepts in general. Actually, their students are trained to solve many problems without deep understanding. As a result, students' mathematical learning achievements are declining. Therefore, mathematical reasoning is greatly important. Considering reasoning as a significant ability is one of the factors affecting the progress of students. The abilities related to students' mathematical literacy include the ability to reason.

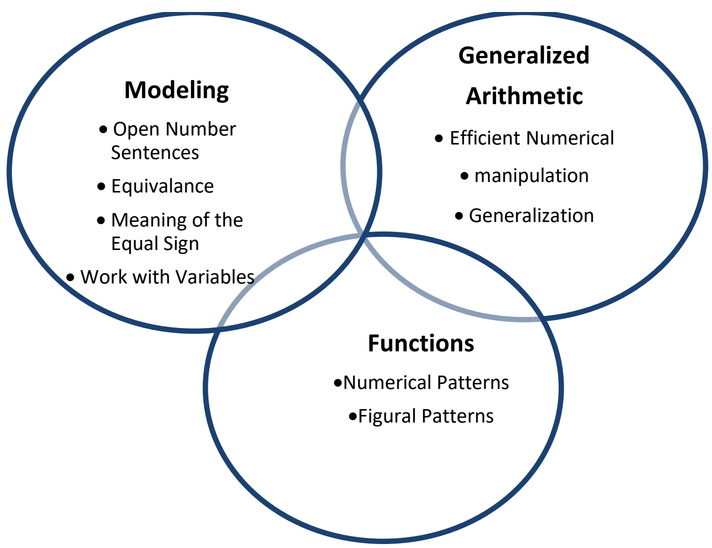

Figure 4. Algebraic thinking model with its components (Ralston, 2013)

As a consequence, the reasoning abilities of students in developing countries should be improved. One of the argumentative topics that has improved significantly is the abstract concept. Most students find it difficult to grasp abstract concepts, especially in algebra. Algebraic reasoning is considered difficult for intermediate students because they are involved in making generalizations through experience with numbers and calculations. Creating such ideas is possible by using

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meaningful symbolic systems and exploring the concepts of patterns and functions. Difficulties in learning algebra can be managed by identifying the underlying reasons related to mathematics. As an alternative to these difficulties, teachers should design a learning plan that develops algebraic reasoning. If students can participate well in algebraic thinking, it helps them in their cognitive process to learn algebra. The process of algebraic reasoning is different for each student. This difference is due to the fact that each person has different unique characteristics. Another issue that may make a variation between people is the gender difference. Men and women indicate their individual differences in various ways such as emotions, behavior, language processes, visual abilities, and mathematical difficulties. Algebraic reasoning process includes observing the pattern or regularity, formulating generalizations and conjectures related to the observed order, evaluating/testing conjectures, constructing and evaluating mathematical reasoning, and describing/confirming logical conclusions about some ideas and their relations (Indraswari et al., 2018).

In recent decades, researchers have provided different opinions on the procedures of teaching and learning algebra. For example, Socas (2011) considered algebraic thinking implicit in elementary school students. The author believes that students through generalization abilities move to higher grades where they have the opportunity to improve algebraic thinking. Some researchers claim that elementary school learners can regard arithmetic operations as functions (one of the components of algebra) (Schliemann et al., 2003), symbolize arithmetical relationships algebraically, perform functional relationships or solve problems utilizing diagrams and tables (Blanton & Kaput, 2005).

Sibgatullin et al. (2022) examined 36 studies in an organized review of algebraic thinking in academic education. They claimed that in-service or pre-service teacher training was necessary for the promotion of algebraic thinking, and unusual exercises like games should be implemented in the classroom. Moreover, they found that teaching strategies such as geometry representation, multiple representation strategies, and mental calculation activity also improved algebraic thinking. Somasundram (2021) explored the role of cognitive factors in 720 fifth-grade students' algebraic thinking. The analysis of structural equation modeling presented that cognitive factor had significant effect on students' algebraic thinking. The most leading factor was symbol comprehension, pattern comprehension, number comprehension, and operation comprehension.

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These findings suggested that educators should apply activities regarding these cognitive factors for teaching mathematics to increase students' transfer from arithmetic to algebra. Kusumaningsih and Herman (2018) studied on improving the students' algebraic thinking using multiple representation strategies in realistic math education. By examining eighth-grade students, they found that there was an interaction between multiple representation strategies using a realistic approach on algebraic thinking ability. Learners who used multiple representation strategies indicated better algebraic thinking ability performance than current scientific learning group. Besides, by applying a realistic approach, more than 75 percent of learners completed learning through multiple representation strategies.

According to the previous reviews and studies, the investigation in the field of algebraic thinking has been conducted based on reasoning and calculation errors. However, a consistent tool or model of algebraic thinking has not been introduced and presented. There is a need to have a standardized tool for a more detailed examination of the efficient components to investigate and evaluate algebraic thinking, which has not been clearly defined in the previous studies.

Material and Methods

The methodology in the research was based on the descriptive strategy and the survey method was conducted to collect data. Using the availability sampling, 20 relatively strong male and female students were selected based on their above-average scores in the fifth-grade mathematics class. The participants were studied in the sixth-grade of 10 public schools in the 5th education district of Tehran in the academic year of 2020-2021. During virtual education – the Coronavirus pandemic- two students were selected from each school. Based on two reliable sources; Somasundram (2018) for the thinking section and Mardiyana (2019) for the reasoning section in the field of algebra; a two-part questionnaire of open and closed-ended questionnaire was designed by researchers (see Appendix A). The Likert scales and their codes in the first part (thinking) were as: the answer was stated without reason or formula, the answer was stated by formula and without reason, no answer was given, the answer was stated by formula, the answer was stated by reason and formula. In the second part of the questionnaire (reasoning), the codes were as: in all cases, the student answers randomly (lack of mastery); in some cases, the student answers with ability and certainty, It is not possible to give a certain opinion, In some cases, the student does not answer with ability and certainty; In all cases, the student answers with full ability and certainty (see Appendix B). Therefore, in both sections in order of options, the coding was specified respectively to codes 1 to 5. In the second part of the questionnaire (reasoning), the researchers applied their opinions based on the answers in class tests and final exams. The investigation was in a routine of eight months of the academic year of 2020-2021 through virtual education in the sixth-grade mathematics class. These opinions were reviewed and confirmed by experienced teachers and experts in mathematics education. At first, the content validity of the questionnaire in terms of the Content Validity Ratio (CVR) index was checked based on the opinions of experts in mathematics education and educational sciences. Specialists with doctorate and master's degrees asked over 10-year experienced teachers in elementary school for their opinions. Based on the results of the four-month survey in this field, the CVR index of the questionnaire showed good content validity for algebraic thinking. After that, according to the answers of 30 sixth-grade students over six months, the reliability of the questionnaire was confirmed based on the results of Cronbach's alpha with a value of 0.78. The aforementioned questionnaire contains two basic sections with sub-sections as follows. In the first section, Thinking includes three factors: the generalized arithmetic factor (factor 1) consists of questions 1 to 6, the modeling factor (factor 2) includes questions 7 to 13, and the functions factor (factor 3) includes questions 14 to 18 that were designed based on Ralston's algebraic thinking model (2013) and the Somasundram test (2018). In the second section, Reasoning is the fourth factor with the sub-factors of following the pattern or order, forming generalization and conjecture about the pattern, evaluation/examination of conjectures and hypotheses, designing and evaluating of mathematical reasoning, describing and confirming logical conclusions in some ideas and their relationship which were adapted from the questionnaire of Mardiyana (2019).

Results and Discussion

In this section, at first, descriptive indices were investigated in the context of examining the components of the algebraic thinking model in terms of mean, maximum frequency, and standard deviation. Then, the quality of the students' answers in terms of algebraic thinking was reviewed. Therefore, in the initial part, data analysis was used by SPSS software version 26. And the TOPSIS

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fuzzy method was used for the explanation and ranking of the components of the algebraic thinking model.

Table 1. Descriptive indices of the components of the algebraic thinking model

Statistical Indices / Components	Mean	Mod	Standard deviation
Generalized arithmetic	2.92	3.17	0.56
Modeling	2.89	3.57	0.54
Functions	2.87	3.20	0.55
Reasoning	2.72	2.73	0.50

As shown in Table 1, the mean of students' answers and the qualitative estimation of answers by researchers in each of the components of algebraic thinking indicated that the highest mean was related to Generalized arithmetic and Modeling. In other words, it was shown that in the relevant questions, the students achieved more generalized arithmetic (2/92) and modeling in algebraic thinking (2/89). And they also tended to answer through generalization and relied on pre-learned formulas. The lowest dispersion of answers was related to the reasoning component. In the following, the one-sample t-test was used for the inferential analysis of the data and the results of which were shown in Table 2:

Table 2. One-sample t-test results of algebraic thinking model components

	value = 4			
	t	Degrees of freedom	*P*-value	Mean difference
Generalized arithmetic	-8.53	19	0.000	-1.07
Modeling	-9.13	19	0.000	-1.12
Functions	-8.98	19	0.000	-1.11
Reasoning	-11.17	19	0.000	-1.27

As it was observed in Table 2, the p-value was less than 0.05. So, it can be claimed with a confidence interval of 95%, in the components of algebraic thinking, students often had either a

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certain opinion (no answer) or they answered in a formulaic way and without sufficient mastery. The biggest mean difference was related to the reasoning component which indicated that students faced challenges in the component.

In order to rank and identify the priorities of the components of the algebraic thinking model, the Fuzzy TOPSIS method was used. Multi-criteria decision-making is one of the approaches that can be used to solve complex problems regarding different areas of human activity, like engineering sciences, social sciences, economics, and management. The Fuzzy TOPSIS as one of the classic compensatory methods in multiple decision-making was presented by Hwang and Yoon (1981) to solve priority problems based on similarity with positive and negative ideal solutions. To use this method, a decision matrix is needed. The rows of the matrix are the options and the columns are the criteria. With a systematic approach, the fuzzy TOPSIS decision-making method can be developed into a fuzzy space. The chosen option of the method should have the shortest distance from the positive ideal (the best possible state, d_i^+), and on the other side has the greatest distance from the negative ideal (the worst possible state, d_i^-). It is more efficient to use the approach, especially when the goal is to solve a decision-making problem in a group. According to the theory, a fuzzy number is a fuzzy set in which, x adopts the actual values of a member of the collection R. The most used fuzzy numbers are triangular and trapezoidal fuzzy numbers. Triangular fuzzy numbers are mostly used due to simple calculations. Therefore, we used triangular fuzzy numbers in the research. The Fuzzy TOPSIS technique introduced by Chen and Hwang (1992) includes the steps listed in Table 3 according to the nature of the questionnaire. It should be noted here that the 'weight' referred to the weight of experts' opinions, which were considered the same. The fuzzy numbers and linguistic expressions used in the Chen (2001) research were shown in Table 3:

Table 3. Fuzzy numbers and their corresponding expressions

Verbal expression	Fuzzy number
Very weak	(1,1,3)
Weak	(1,3,5)
Medium	(3,5,7)
Good	(5,7,9)
Very good	(7,9,11)

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The First Stage: suppose that the fuzzy decision matrix of people's opinions is as follows:

$$\widetilde{D} = \begin{bmatrix} \widetilde{x_{11}} & \widetilde{x_{12}} \cdots & \widetilde{x_{1n}} \\ \widetilde{x_{21}} & \widetilde{x_{22}} \cdots & \widetilde{x_{2n}} \\ \widetilde{x_{m1}} & \widetilde{x_{m2}} \cdots & \widetilde{x_{mn}} \end{bmatrix} \quad (1)$$

$$\widetilde{W} = [\widetilde{W_1}, \widetilde{W_2}, \ldots \widetilde{W_n}] \quad (2)$$

In the matrix, i is the number of examined components (m); j is the number of samples (n), $\widetilde{X_{ij}}$ is the opinion of i person about the component of j th, which is calculated as the following fuzzy number:

$$\tilde{X} = (a_{ij}, b_{ij}, c_{ij}) \quad (3)$$

W_{ij} is the significance of the opinions of each person that is stated as a fuzzy number.

The Second Stage: unscaled the decision-making matrix:

In this stage, the fuzzy decision-making matrix of people's opinions should be converted into a fuzzy unscaled matrix$(\tilde{R})$. To obtain the matrix $\tilde{R}$, we should use the following relations:

$$\tilde{R} = [\tilde{r}_{ij}]_{m\times n} \quad i = 1,2,\ldots,m \quad j = 1,2,\ldots,n \quad (4)$$

In the relation, m is the number of options and n is the number of respondents. If the fuzzy numbers are in the form of (a, b, c), $\tilde{R}$ which is the unscaled matrix (normalized) is obtained as follows:

$$\tilde{r}_{ij} = \left(\frac{a_{ij}}{c_j^*}, \frac{b_{ij}}{c_j^*}, \frac{c_{ij}}{c_j^*}\right) \quad (5)$$

In the relation, c_j^* is the maximum value of c in j th expert among all the options. The following relation expresses the issue:

$$c_j^* = \max_i c_{ij} \quad (6)$$

The Third Stage: creating the fuzzy weight unscaled matrix ($\tilde{v}$): assuming the vector $\widetilde{W_{ij}}$ as an input to the algorithm. The relation is:

$$\tilde{V} = [\tilde{v}_{ij}]_{m\times n} \quad i = 1,2,\ldots,m \quad j = 1,2,\ldots,n \quad (7)$$

$$\tilde{v}_{ij} = \tilde{r}_{ij} \otimes \tilde{w}_j$$

In the relation, $\widetilde{r_{ij}}$ is the obtained unscaled matrix from the second stage.

The Fourth Stage: specifying the fuzzy positive ideal (FPIS, A^+), and fuzzy negative ideal (FPIS, A^-):

$$A^+ = (v_1^*, v_2^*, ..., v_n^*) \qquad A^- = (v_1^-, v_2^-, ..., v_n^-) \tag{8}$$

Here, the fuzzy positive ideal value and fuzzy negative ideal value which were introduced by Chen and Hwang (1992) are used. These values are:

$$v_j^* = (1,1,1) \qquad v_j^- = (0,0,0) \tag{9}$$

The Fifth stage: calculating the sum of the distances of each option from the fuzzy positive ideal and the fuzzy negative ideal:

If $\tilde{A}$ and $\tilde{B}$ are two fuzzy numbers as follows, then the distance between these two fuzzy numbers is obtained by the following relation:

$$\tilde{A} = (a_1, b_1, c_1)$$

$$\tilde{B} = (a_2, b_2, c_2)$$

$$D(\tilde{A}, \tilde{B}) = \sqrt{\frac{1}{3}[(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2]} \tag{10}$$

According to the above explanation about how to calculate the distance between two fuzzy numbers, we get the distance of each component from the positive ideal and the negative ideal:

$$d_i^* = \sum_{j=1}^{n} d(\tilde{v}_{ij} - \tilde{v}_{ij}^*) \quad i = 1,2,...,m \tag{11}$$

(12)

$$d_i^- = \sum_{j=1}^{n} d(\tilde{v}_{ij} - \tilde{v}_{ij}^-) \quad i = 1,2,...,m$$

The Sixth stage: calculating the relative proximity of the *i* th option to the ideal solution. This relative proximity is defined as follows:

$$CC_i = \frac{d_i^-}{d_i^* + d_i^-} \quad i = 1,2,...,m \tag{13}$$

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The Seventh step: ranking the options: in descending order of CC *i*, we can rank the options in the problem.

According to the purpose of the research to rank the components of algebraic thinking, we reviewed the participants' answers. And after going through the above steps, the last two results were obtained as follows.

Table 4. Sample results of positive and negative ideals of algebraic thinking components

Student	Positive ideal (di^+)				Negative ideal (di^-)			
	Component 1	Component 2	Component 3	Component 4	Factor 1	Factor 2	Factor 3	Factor 4
Student 1	1.34	1.34	1.34	1.26	0.14	0.14	0.14	0.16
Student 2	101.11	1.27	1.44	1.44	8.16	0.15	0.10	0.10
Student 3	100.94	101.11	101.11	100.94	8.16	8.16	8.16	8.16
Student 4	1.34	1.34	1.34	1.34	0.14	0.14	0.14	0.14
Student 5	100.94	101.11	101.11	100.94	8.16	8.16	8.16	8.16
Student 6	100.83	101.11	100.83	100.83	8.16	8.16	8.16	8.16
Student 7	100.83	100.83	100.83	100.83	8.16	8.16	8.16	8.16
Student 8	101.00	100.83	100.83	100.83	8.16	8.20	8.16	8.16
Student 9	100.83	100.83	100.83	100.83	8.16	8.16	8.16	8.16
Student 10	100.83	100.83	100.83	100.83	8.16	8.16	8.16	8.16
Student 11	1.36	1.36	1.23	1.36	0.12	0.12	0.12	0.12
Student 12	1.23	101.00	1.23	101.00	0.16	8.16	0.16	8.16
Student 13	1.23	1.50	1.36	1.36	0.16	0.08	0.12	0.12
Student 14	1.23	1.36	1.36	1.36	0.16	0.12	0.12	0.12
Student 15	1.23	1.23	1.36	1.36	0.16	0.16	0.12	0.12
Student 16	1.36	1.23	1.36	1.36	0.12	0.16	0.12	0.12
Student 17	1.36	1.23	1.36	1.36	0.12	0.16	0.12	0.12
Student 18	1.36	1.23	1.36	1.36	0.12	0.16	0.12	0.12
Student 19	1.27	1.27	1.27	1.27	0.15	0.15	0.15	0.15
Student 20	1.36	1.23	1.36	1.50	0.12	0.16	0.12	0.08

The calculation of positive and negative ideals was evaluated in Table 4 after calculating the maximum matrix, fuzzy unscaled matrix, and fuzzy weight unscaled matrix. This is the stage where we can prepare the weight index (CC*i*) for ranking the four components of the algebraic thinking model.

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Table 5. Relative proximity results of CC*i* (weight)

Components of algebraic thinking model	The distance to the positive ideal ($di+$)	The distance to the negative ideal ($di-$)	Relative proximity (weight) (CCi)	Rank
Generalized arithmetic	822.95	66.92	0/075201	second
Modeling	823.22	66.99	0/075251	first
Functions	723.71	58.78	0.075119	third
Reasoning	823.33	66.72	0.074962	fourth

The results of the investigations in the seven stages were explained. After examining the fuzzy unscaled matrix and the fuzzy weighted unscaled matrix, the ideal positive and negative fuzzy matrices were formed in Table 4 based on the algebraic thinking components of 20 students. Then, to weigh and rank the components based on relative proximity in Table 5, it was observed that the modeling component rank was first (0.075251). And the reasoning component was ranked fourth (0.074962). The third and fourth ranks belonged to the components of functions and reasoning, respectively. In the following, we indicated a sample of students' handwriting in the three components of algebraic thinking as an example:

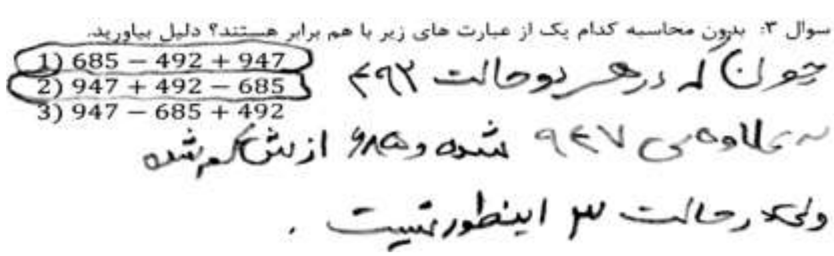

Figure 5. A sample of a student's handwriting in question 3 (see Appendix A) related to the arithmetic component. The student drew a line around options 1 and 2, and explained that "Because in both options, 492 is added to 947, and 685 is subtracted from it, but it is not the case in option three."

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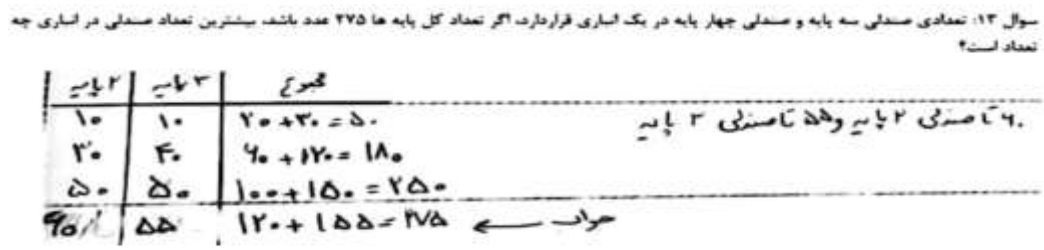

Figure 6. A sample of the student's handwriting in question 13 (see Appendix A) related to the modeling component. To get the maximum number of chairs by trial and error, the student counted the number of three and four-legged chairs in different positions.

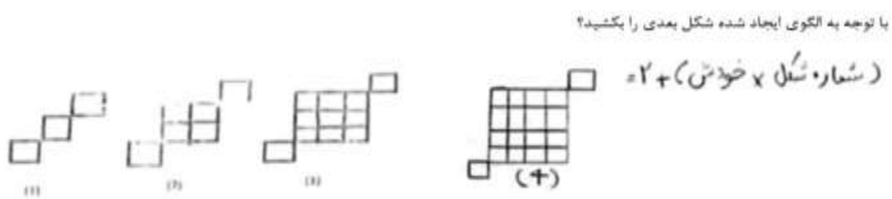

Figure 7. A sample of a student's handwriting in question 18 (see Appendix A) related to the function component. The student wrote the following formula to answer the question. (Shape number× Shape number)+2=)

Conclusion

Education and curricula should be designed to improve students' knowledge along with educational developments. Teaching mathematical concepts in elementary school can significantly influence the future professional life of students to enter university, their employment, etc. Mathematics consists of three basic parts: arithmetic, algebra and geometry. According to these categories, the main focus in the current study was on algebraic thinking of elementary students. To promote students' algebraic thinking, instructors should work on communication, development, and knowledge/skills in teaching algebra. They should support students to take part in higher levels of mathematical thinking and develop problem-solving skills. Such efforts in the direction of developing algebraic thinking have always been discussed as a challenge during school education. In the study, we attempted to focus on algebraic thinking and reviewing the literature, to address the components of algebraic thinking, and to examine other components such as reasoning in the category. Since the study was conducted during virtual training in Coronavirus pandemic, it exposed certain limitations and conditions. The research tool was designed and confirmed to measure and rank the components of algebraic thinking, namely, generalized arithmetic, modeling,

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function, and reasoning. And as a result, we could recognize the priorities and problems of students in algebraic thinking.

As noted earlier, the elementary course is one of the basic and effective procedures in arithmetic and algebra learning for higher levels of education. Moreover, teachers does not address algebraic thinking sufficiently and deeply for the first to sixth grades students in elementary school. Therefore, having a tool to evaluate the category would facilitate elementary school teachers' obligations. Accordingly, the questionnaire was reviewed and implemented to collect data regarding the study's purposes. The findings was examined and the mean of the students' answers was checked. The answers were qualitatively estimated in each of the components of algebraic thinking and their ranking through TOPSIS Fuzzy. The outcomes indicated that in the mentioned components, the highest mean was related to generalized arithmetic and modeling. In other words, the students achieved more generalized arithmetic and modeling in algebraic thinking in the relevant questions, and they tended to answer through generalization and relied on pre-learned formulas. It was observed that the modeling component ranked first and the reasoning component ranked fourth. In the course of virtual education, the modeling component and generalized arithmetic were ranked first and second in the evaluation. The reason was that from the first grade of elementary school, the students encountered a consistent model and organized findings of expressions. Thus, they tried to follow and take patterns to reach the answer. In generalized arithmetic, students in first-grade of elementary school mostly imagined shapes hypothetically with tangible manipulations and they did calculations according to such a pattern. In the functions component which was ranked third, the students practiced and repeated the component in algebraic thinking mostly in pattern-finding and diagrammatic patterns. In the reasoning component, the students' writings and answers in the algebra sections of class tests were estimated. It was found that the students could not give reasons, express reasons, and criticize expressions and algebraic expressions. Therefore, the component was ranked fourth.

Therefore, it can be inferred that algebraic thinking is a transformation of an abstract approach to quantitative situations that focuses on general relational aspects that are not necessarily symbolic letters. But in the end, they can be used as cognitive support for the introduction and interpretation of mathematical context in school. Algebraic thinking involves strong symbolization and generalization. Algebraic thinking ability starts when a person can use a specific number to reason

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about a mathematical expression. It implies using representations and making relationships in meaningful ways, and as such, focuses more on relationships between numbers and generalized ideas. Activities during teaching, the development of mathematical concepts, and the development of teaching strategies have an impact on students' algebraic thinking. Modeling assignments have a positive effect on students' algebraic thinking and can improve the components of algebraic thinking. Also, the quality of understanding numbers, operations, symbols, non-numerical quantities, positive and negative integers, and rational numbers leads to algebraic thinking development. In addition, multiple representation strategies, mental calculation activities, symmetry, and other practical activities of pattern finding influence on the algebraic thinking of elementary school students. When the learning environment is ready for implicit operations/expressions of algebra, students can experience algebraic thinking from the beginning of elementary school. Students with high-level algebra skills are those who have high-level math skills. Algebraic thinking ability in students can lead to problem-solving in class. It appears from the aforementioned investigations that valid and reliable measurement tools have not been developed to determine algebraic thinking. To accomplish the aim, we attempted to provide a valid tool to such an extent that the components of algebraic thinking in four categories can be explained and prioritized in detail. The issue of prioritizing the components of algebraic thinking can be effective for teaching in mathematics classes. More information on determining the challenges of students in the field of algebra would help teachers to establish a greater degree of accuracy on this matter.

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Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Ethics statement

The studies involving human participants were reviewed and approved by ethics committee of Islamic Azad University.

Author contributions

All authors contributed to the study conception and design, material preparation, data collection and analysis. All authors contributed to the article and approved the submitted version.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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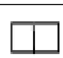
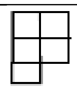
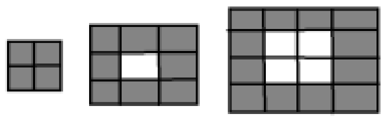
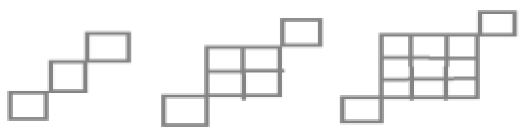
Appendix A

<table>
<tr><td rowspan="6">Generalized Arithmetic (Factor 1)</td><td>Question 1: The sum of two odd numbers is always even. Explain whether the statement is correct or incorrect.</td></tr>
<tr><td>Question 2: A frog climbs a staircase. One of the numbers 1 and 2 is written on each step. The frog must jump up equal to the number written on that step (for example, if the number of steps is 2, the frog jumps up two steps or if the number is 1, it jumps up only one step). If the frog started from the first step and reached the 100th step, it means that the frog climbed a hundred stairs. Which number of stairs were in front of the frog to jump up and pass a total of one hundred stairs?
Write all possible conditions. How do we sure that we have found all possible conditions?</td></tr>
<tr><td>Question 3: Without calculation, which of the following expressions are equal? Give a reason.
1) 685−492+947
2) 947+492−685
3) 947−685+492</td></tr>
<tr><td>Question 4: Without numerical calculation, put the appropriate number in the blank (explain).
(458+⋯)+(856−122)=458+856</td></tr>
<tr><td>Question 5: Without numerical calculation, put the appropriate number in the blank (explain).
(8×36)+36=⋯×36</td></tr>
<tr><td>Question 6: Ali intends to buy educational books, notebooks, and pens in a stationery store. The price of each notebook is half of the price of the educational book, and the price of each pen is a quarter of the price of each notebook. Complete the table below.
<table>
<tr><th>Educational book</th><th>Notebook</th><th>Pen</th></tr>
<tr><td></td><td>8000</td><td></td></tr>
<tr><td></td><td></td><td>1000</td></tr>
<tr><td>6000</td><td></td><td></td></tr>
</table></td></tr>
<tr><td rowspan="7">Modeling (factor 2)</td><td>Question 7: Get a number that if we multiply by 6 and add it to 12, we get the same answer as if we multiply the same number by 9.</td></tr>
<tr><td>Question 8: Without numerical calculation, put a suitable number in the blank (explain).
24+12= …×3</td></tr>
<tr><td>Question 9: The sum of the grades of 8 subjects of a student is 12.5. We want to add some points to the two lower subjects, so that the sum of the grades becomes 15. If we add 1 point to one of the subjects, how many points should be added to the second one?</td></tr>
<tr><td>Question 10: The perimeter of the square is equal to 24, by calculating the length of each side of the square, find the area of the square.</td></tr>
<tr><td>Question 11: A rectangle with a perimeter of 320 meters is assumed. If the length is seven times the value of the width, find the area of the rectangle.</td></tr>
<tr><td>Question 12: We have two nested rectangular cubes with a square base, the side length of the base of the smaller rectangular cube is 70% of the side length of the base of the larger rectangular cube. If the length of the larger cube is 10 cm and the height of the larger cube is 20 cm, we pour some water in the space between these two rectangular cubes. If we remove the big rectangular cube, how will the height of the water change?</td></tr>
<tr><td>Question 13: There are some three-legged chairs and four-legged chairs in a warehouse. If the total number of legs is 275, what is the maximum number of three-legged chairs in the warehouse?</td></tr>
<tr><td></td><td>Question 14: If  is equal to $\frac{3}{4}$, in this case, the figure below is equal to what fraction? Why?</td></tr>
</table>

[DOI: 10.22034/6.4.57]

<table>
<tr><td rowspan="5">Functions (factor 3)</td><td colspan="2"></td></tr>
<tr><td colspan="2">Question 15: A person plans to broadcast an instant message among 999 of her friends. All of these 1000 people (the person and her friends) have their contact numbers. If a person can inform two people after every minute, how many minutes are needed for them to be informed of the news?</td></tr>
<tr><td colspan="2">Question 16: Consider the following pattern. How many colored squares does the 250th figure have? Why?
</td></tr>
<tr><td colspan="2">Question 17: Put the right number in the blank with a reason.
2,8,18,32,…</td></tr>
<tr><td colspan="2">Question 18: According to the created pattern, draw the next one.

</td></tr>
<tr><td rowspan="11">Reasoning (factor 4)</td><td rowspan="2">Following the pattern or order</td><td>Question 19: Does the student have the ability to determine the changes that occurred in a row of numbers or patterns?</td></tr>
<tr><td>Question 20: Does the student have the ability to determine the types of patterns in a row of numbers and shapes?</td></tr>
<tr><td rowspan="2">Forming generalization and conjecture about patterns</td><td>Question 21: Does the student have the ability to continue the sequence of numbers to the next numbers or patterns?</td></tr>
<tr><td>Question 22: Does the student have the ability to create formulas for sequences/patterns/numbers?</td></tr>
<tr><td rowspan="3">Evaluation/examination of conjectures and hypotheses</td><td>Question 23: Does the student have the ability to solve problems using mathematical formulas in algebraic situations?</td></tr>
<tr><td>Question 24: Does the student have the ability to determine the quantity of a sequence of numbers?</td></tr>
<tr><td>Question 25: Does the student have the ability to do mathematical operations related to composition in different algebraic expressions?</td></tr>
<tr><td rowspan="2">Designing and evaluating of mathematical reasoning</td><td>Question 26: Does the student have the ability to analyze the answers of an algebraic expression?</td></tr>
<tr><td>Question 27: Does the student have the ability to determine the reasons related to a combination or do an operation in an algebraic expression?</td></tr>
<tr><td rowspan="2">Describing and confirming logical conclusions in some ideas and their relations</td><td>Question 28: Does the student have the ability to determine the result related to the sequence of numbers/patterns and algebraic expressions?</td></tr>
<tr><td>Question 29: Does the student have the ability to conclude in various problems of any algebra field?</td></tr>
</table>

[DOI: 10.22034/6.4.57]

Appendix B

Coding students' answers for factor 1 to 3

The fifth option: the answer was stated by reason and formula. Code 5
The fourth option: the answer was stated by formula. Code 4
The third option: no answer was given. Code 3
The second option: the answer was stated by formula and without reason. Code 2
The first option: the answer was stated without reason or formula. Code 1

Coding students' answers for factor 4

The fifth option: In all cases, the student answers with full ability and certainty. Code 5
The fourth option: In some cases, the student does not answer with ability and certainty. Code 4
The third option: It is impossible to give a certain opinion. Code 3
The second option: In some cases, the student answers with ability and certainty. Code 2
The first option: In all cases, the student answers randomly (lack of mastery). Code 1

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